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OPTIMAL PATHS OF CAPITAL ACCUMULATION UNDER THE MINIMUM TIME OBJECTIVE

BY
MORDECAI KURZ

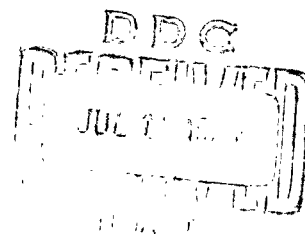
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INSTITUTE FOR MATHEMATICAL STUDIES IN THE SOCIAL SCIENCES
Applied Mathematics and Statistics Laboratories
STANFORD UNIVERSITY
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Introduction

The important question of investment criteria and optimal strategy of development has been discussed by various authors from many different points of view. In principle, a distinction can be made between "competitive" growth paths and "non-competitive" paths. In "competitive" paths we mean the growth path which is determined by a model in which decentralized structure of production and consumption is postulated; producers' expectation functions and profit maximization leading to demand function for capital is explicitly introduced; and consumers' preferences leading to a savings function completes the model.

Once we depart from the deterministic competitive model, it becomes very unclear how much the economist can say about social development goals. If we allow the preference function of some planning board to determine the optimal development strategy, we are, in fact, introducing political and military considerations, national pride, past experience of the nation and other sociological factors that mold the objectives of the nation.

It is quite clear that the setting of objective functions for long-range economic development will always be a disputable question and the

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only hope a writer has is to be able to give reasonable justification for any choice of an objective function.

The present work is, in a certain sense (to be explained in the last section), a continuation of Srinivasan's work [12] within the context of a two sector type economy discussed by Drandakis [3], Kurz [4], Meade [5], Srinivasan [13], Uzawa [15] and others. The model represents a closed economy in which foreign trade, government fiscal policy and technological change are excluded. The inclusion of foreign trade and technological changes are natural extensions of this work, while government policy is implicitly assumed in the statement of the objectives of the economy.

In Section (A) we discuss the general nature of our objective function. In Section (B) the model is presented and the problem is explicitly stated. Section (C) contains the main body of analysis while Section (D) is devoted to a comment.

(A) Social Goals

The recent discussions regarding social goals have pointed out two main differences between preferences of individuals and society as a whole. These are: (1) Society, contrary to the individual, has an infinite horizon. (2) Society should not discriminate among generations; hence, it should not have a discount factor. These requirements have caused substantial logical difficulties indicated in [1], [14], and others.

In trying to accommodate requirements (1) and (2) above, let us consider the following point of view. It stands to reason that society should seek the highest technologically feasible level of consumption

per capita for its citizens. The need not to discriminate among generations imposes a restriction upon such a goal. This restriction is that the maximum level to be obtained should be a sustainable level. A sustainable rate of consumption per capita is a rate that the economy can attain and maintain forever with the appropriate investment policy. Social goals may, therefore, be defined in terms of such sustainable rates of consumption per capita. From the point of view of consumption theory, our suggested point of view is some kind of "permanent income hypothesis" for society as a whole. The nature of the maximum sustainable rate, called the maximum terminal path level, was discussed recently in [2], [6], [7], [8], [9], [11].

In a certain sense, the attainment of such maximal terminal path level for consumption per capita may be regarded as reaching "economic maturity" for an underdeveloped economy. Hence, if we impose upon our underdeveloped economy the goal of reaching this maximal terminal path, there arise two natural questions: (1) Since attainment of this von Neumann path takes time, how does society feel about the transition period and the level of consumption during this period? (2) Once a society is on the path, a certain intrageneration agreement should exist to maintain the optimal investment policy forever.^{1/}

Question (1) above is the serious one and we shall devote some thoughts to it below. If one reads, however, the vast economic-political literature concerning the problems of development, one can notice that everywhere politicians and economists state their goal as temporary restriction of consumption (politically feasible) so as "to reach economic maturity as soon as possible." The theoretical

implications of such social goals can be stated as follows: Given certain institutional restrictions on consumption, try to reach the maximum terminal path in minimum time. In "reaching" the path we mean that the composition of the capital stocks and labor should be such that when we reach the path we stay on it indefinitely. Obviously many arguments can be raised against such social goal function. It assigns no benefits to periods in which consumption per capita is temporarily above the maximum terminal path level; or negative benefits to periods in which we are below this path. However, it seems interesting enough to investigate the implications of such an objective and study the structure of the optimal strategy. On the basis of the results obtained from the pure minimum time problem, we shall be able to evaluate some general principles of development strategy and suggest natural extension of our approach.

(B) The Model

In the analysis that follows, we shall work with a two-sector economy. The first sector uses capital and labor, and produces capital goods; the second sector produces consumer goods and uses capital and labor. More specifically, let

- K_1 be capital goods employed in the capital goods sector
- L_1 be labor employed in the capital goods sector
- K_2 be capital employed in the consumer goods sector
- L_2 be labor employed in the consumer goods sector

Then the production functions are

$$(1) \quad Q_i = F_i(K_i, L_i), \quad i = 1, 2,$$

and we shall assume

(1a) F_i are homogeneous of first degree and concave; hence,

$$F_i(K_i, L_i) = L_i f_i\left(\frac{K_i}{L_i}\right), \quad f_i > 0, \quad f_i'' < 0, \quad i = 1, 2.$$

$$(1b) \quad \begin{aligned} f_i(0) &= 0, \quad f_i'(0) = \infty \\ f_i(\infty) &= \infty, \quad f_i'(\infty) = 0. \end{aligned}$$

(2) As for capital goods, we shall assume that capital goods are not "shiftable" after they have been placed in either sector. In other words, new capital goods can be invested in either sector but after being installed they cannot be moved from one sector to the other. We retain the assumption that capital-labor substitution prevails within each sector, but only labor can move freely from one sector to the other. Our basic differential equation system is formulated as follows:

Let L be total labor supply,

$$(2a) \quad k_1 = \frac{K_1}{L},$$

$$(2b) \quad k_2 = \frac{K_2}{L},$$

and we shall assume

(3) $L = e^{nt}$ so that the labor force grows at a constant rate. Capital goods depreciate at a constant proportional rate μ .

Since

$\dot{K}_1 + \mu K_1$ is total gross investment in the capital goods sector,

and

$\dot{K}_2 + \mu K_2$ is total gross investment in the consumer goods sector,
then

$$(3a) \quad F_1 = (\dot{K}_1 + \mu K_1) + (\dot{K}_2 + \mu K_2) .$$

Let

$$(3b) \quad \frac{\dot{K}_1 + \mu K_1}{F_1} = u$$

and

$$(3c) \quad \frac{\dot{L}_1}{L} = v .$$

Then, from the definition of k_1 and k_2 , we have

$$\begin{aligned} \dot{k}_1 &= \frac{\dot{K}_1}{K_1} k_1 - \frac{\dot{L}}{L} k_1 \\ \dot{k}_2 &= \frac{\dot{K}_2}{K_2} k_2 - \frac{\dot{L}}{L} k_2 . \end{aligned}$$

Then, using (3a) and the notation of (3b) and (3c), we obtain

$$(4) \quad \begin{cases} (4a) & \dot{k}_1 = uv f_1\left(\frac{k_1}{v}\right) - (\mu + n)k_1 \\ (4b) & \dot{k}_2 = (1 - u)v f_1\left(\frac{k_1}{v}\right) - (\mu + n)k_2 . \end{cases}$$

The system (4) is the fundamental system of differential equations. If we start with some initial $k_1(0)$ and $k_2(0)$, and then choose arbitrary $u(t)$, $v(t)$, the system (4) can be solved for the implied paths of $k_1(t)$ and $k_2(t)$. Note that $u(t)$, being the proportion of new investments placed in the capital goods sector is a variable that can vary between 0 and 1 and, hence, if $u = 0$, all new capital goods are invested in the consumer goods sector, while if $u = 1$, all new capital

goods are invested in the capital goods sector. Similarly, if $v = 0$, then all labor is employed in the consumer goods sector, and if $v = 1$, all labor is employed in the capital goods sector. We may also note that $v = 0$ and the concavity of f_1 imply that $\lim_{v \rightarrow 0} v f_1\left(\frac{k_1}{v}\right) \geq 0$.

If $v f_1\left(\frac{k_1}{v}\right)\Big|_{v=0} = 0$, then, by convention, we set $u = 0$. Hence, we

shall assume

$$(5) \quad v = 0 \Rightarrow \begin{cases} u \geq 0 & \text{if } v f_1\left(\frac{k_1}{v}\right)\Big|_{v=0} \geq 0 \\ u = 0 & \text{if } v f_1\left(\frac{k_1}{v}\right)\Big|_{v=0} = 0 . \end{cases}$$

Having the free choice of the composition of new investments (u) and the composition of the labor force (v), our economic problem starts. An objective function is given to us. Let us denote our objective function W , which is $W = W(t, k_1, k_2, u, v)$, and our problem is to maximize

$$(6) \quad D = \int_0^T W(t, k_1, k_2, u, v) dt ,$$

subject to the system (4) and $0 \leq u \leq 1$, $0 \leq v \leq 1$. In other words, we want to choose $u^*(t)$ and $v^*(t)$ such that if we solve (4) and insert in (6), the resulting D^* is maximized.

As we suggested in Section (A) above, our objective will be to minimize the time to economic maturity. In this case, $W \equiv 1$ and $D = T$; hence, we want to minimize T , the time needed to reach the maximum sustainable level of terminal path consumption per capita.

(B.1) The Characterization of the Maximum Terminal Path

We shall present now the precise definition of the maximal

terminal path level. This subject has been discussed in the literature; hence, we shall state without proof the following: Let $v = \frac{L_1}{L}$. Then the maximum terminal path value of consumption per capita is determined by k_1^T , k_2^T , and v^T . These are defined by the following conditions:

$$(6a) \quad v^T = \frac{k_2^T}{1 - v^T} \frac{f_2'}{f_2}$$

$$(6b) \quad (n + \mu) = f_1'$$

$$(6c) \quad \frac{f_1}{f_1'} - \frac{k_1^T}{v^T} = \frac{f_2}{f_2'} - \frac{k_2^T}{1 - v^T}$$

where $f_1 = f_1\left(\frac{k_1^T}{v^T}\right)$, $f_2 = f_2\left(\frac{k_2^T}{1 - v^T}\right)$ and so are the derivatives f_1' and f_2' . By virtue of (1.a) and (1.b), it is true that $0 < v^T < 1$ and $k_1^T > 0$, $k_2^T > 0$. In the plane (k_1, k_2) , the point (k_1^T, k_2^T) has certain properties that are important. These will be proved now.

Theorem 1: Let (k_1^T, k_2^T) be the maximal terminal path values of k_1 and k_2 , then

$$(1) \quad k_2^T < \frac{f_1(k_1^T)}{n + \mu}$$

$$(2) \quad k_1^T < \frac{f_1(k_1^T)}{n + \mu}.$$

Proof: Insert (6.a), (6.b) into (6.c) to obtain

$$\frac{v^T f_1\left(\frac{k_1^T}{v^T}\right)}{n + \mu} = k_1^T + k_2^T.$$

Since $k_1^T > 0$, $k_2^T > 0$ we clearly have

$$k_1^T < \frac{v^T f_1\left(\frac{k_1^T}{v^T}\right)}{n+\mu},$$

but since $0 < v^T < 1$ and by concavity of f_1 , it follows that

$$f_1(k_1^T) > v^T f_1\left(\frac{k_1^T}{v^T}\right), \text{ hence,}$$

$$k_1^T < \frac{f_1(k_1^T)}{n+\mu}$$

$$k_2^T < \frac{f_1(k_1^T)}{n+\mu}.$$

In diagrammatic form, we have:

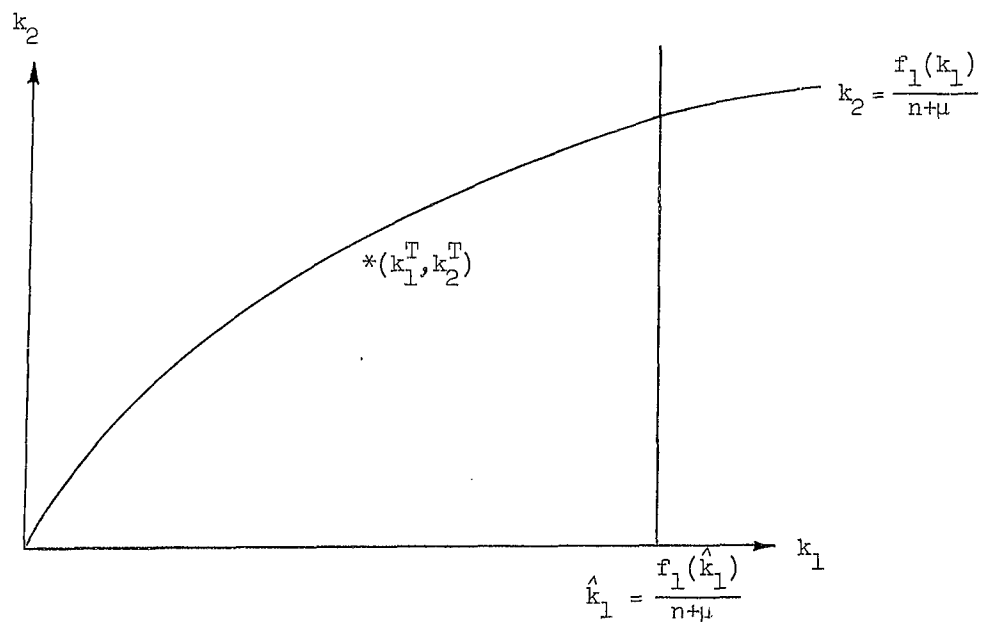


Diagram 1

In most of this paper we shall work in the (k_1, k_2) plane rather than in the two separate (k_1, t) , (k_2, t) planes.

Now, given $k_1(0)$, $k_2(0)$, and the targets k_1^T , k_2^T , our minimum time problem is: choose $u^*(t)$ and $v^*(t)$ in such a way that the implied solutions $k_1^*(t)$, $k_2^*(t)$ (implied by the differential equation (4) above) will have the following properties: (1) $k_1^*(0) = k_1(0)$, $k_2^*(0) = k_2(0)$; (2) $k_1^*(T) = k_1^T$, $k_2^*(T) = k_2^T$; (3) T will be the minimum among all feasible T 's that satisfy (1) and (2).

This form of our problem allows us to utilize the powerful Pontryagin Maximum Principle [10], which will be the basic mathematical tool in what follows.

(G) The Solution of the Minimum Time Problem

As the problem was formulated, the Pontryagin Principle can be applied for the minimum time problem. The conditions of the theorem can be stated as follows:

Let

$$(7) \quad H = \psi_1(u(t)v(t) f_1(\frac{k_1}{v}) - (n+\mu)k_1) + \psi_2((1-u(t))v(t) f_1(\frac{k_1}{v}) - (n+\mu)k_2).$$

Then, if $u^*(t)$ and $v^*(t)$ are the optimal policies, there exist continuous functions $\psi_1(t)$ and $\psi_2(t)$, not both identical to 0,

such that

$$(8a) \quad \frac{d\psi_1(t)}{dt} = -[u^*(t) f_1'(\frac{k_1}{v^*}) - (n+\mu)] \psi_1(t) - [(1-u^*)f_1'(\frac{k_1}{v^*})] \psi_2(t),$$

$$(8b) \quad \frac{d\psi_2(t)}{dt} = (n+\mu)\psi_2,$$

and

(I) for all t , $0 \leq t \leq T$, the function $H(\psi_1, \psi_2, k_1, k_2, u, v)$ as a function of u and v attains its maximum at v^* and u^* .

(II) $\sup_{u,v} H(\psi_1, \psi_2, k_1, k_2, u, v) \geq 0$ for all $0 \leq t \leq T$.

($\sup_{u,v} H(\psi_1, \psi_2, k_1, k_2, u, v)$ is a constant.)^{2/}

Our investigation has to start with the time functions $\psi_1(t)$ and $\psi_2(t)$. Solving the differential equations (8a, 8b) we see that there are two possible solutions for arbitrary $v(t)$ and $u(t)$:

$$(a) \quad \psi_1(t) = B e^{-\int_0^t [u(\tau) f_1'(\frac{k_1}{v}) - (n+\mu)] d\tau},$$

$$\psi_2(t) \equiv 0,$$

$$(b) \quad \psi_1(t) = e^{-\int_0^t g(\tau) d\tau} \left[- \int_0^t e^{\int_0^\tau g(z) dz} A e^{(n+\mu)\tau} [(1-u)f_1'(\frac{k_1}{v})] d\tau + C \right],$$

$$\psi_2(t) = A e^{(n+\mu)t},$$

where A, B, C are constants, $g(\tau) = u(\tau) f_1'(\frac{k_1}{v}) - (n+\mu)$ and $k_1(\tau)$ is the corresponding solution to (4a, 4b).

The Pontryagin theorem gives us essentially a set of consistency conditions. To prove that in fact some u^*, v^* are optimal, we have to find $\psi_1(t)$ and $\psi_2(t)$ continuous such that the consistency conditions are met. It is obvious from the start that the initial conditions are crucial. For different initial conditions, we can expect different optimal policies. The initial conditions will determine both the signs and the sizes of A, B , and C , and also the corresponding optimal policies, v^* and u^* .

Note, however, that whenever solution type (b) holds with $\psi_1 \neq 0$, $\psi_2 \neq 0$, $\psi_1 \neq \psi_2$, then the optimal policy will take the form $v^* = 0$ or $v^* = 1$, $u^* = 0$, or $u^* = 1$. Only when $\psi_1 \equiv 0$ or $\psi_2 \equiv 0$ (not both) is it possible to have different solutions than these two.^{3/}

The complexity of the problem arises from the fact that ψ_1 and ψ_2 as solutions of (8a) and (8b) depend upon v^* and u^* , while in evaluating $H(t)$ and the optimal policy, they determine u^* and v^* .

The economic meaning of this search for a consistent set of conditions is the old search for a set of (shadow) prices, such that behavior under them follows some optimal properties.

In presenting our solutions we shall present in the following section a diagram and description of the optimal policy. In Section C.2, a rigorous proof will be given.

(C.1) Description of the Optimal Strategy

In the plane (k_1, k_2) there are three distinct regions denoted region I, II, and III. These three regions are determined by three curves which are, in turn, defined in terms of the three "extreme policies." Let us define these first.

(a) The $u^* = 1$, $v^* = 1$ policy is the policy according to which all labor is employed in the capital goods sector ($v^* = 1$), and all new capital goods are reinvested in the capital goods sector.

(b) The $u^* = 0$, $v^* = 1$ policy is the policy according to which all labor is employed in the capital goods sector ($v^* = 1$) but all new capital goods are invested in the consumer goods sector ($u^* = 0$).

(c) The $u^* = 0, v^* = 0$ policy is the policy according to which all labor is employed in the consumer goods sector and no new capital goods are built.

The three curves that define regions I, II, and III are the following (see Diagram 2):

(a) The curve DG has the property that it is the collection of all $(k_1(0), k_2(0))$ such that if the $v^* = 1, u^* = 1$ policy is followed, the path will pass through the point (k_1^T, k_2^T) . As for all $v^* = 1, u^* = 1$ policies, the coordinates tend as follows: $k_2(\infty) = 0$ and $k_1(\infty) = \hat{k}_1$ where $\hat{k}_1 = \frac{f_1(\hat{k}_1)}{n+\mu}$.

(b) The curve EF is the collection of all $(k_1(0), k_2(0))$ such that if the $v^* = 1, u^* = 0$ policy is followed, $(k_1(t), k_2(t))$ will pass through (k_1^T, k_2^T) but ultimately will turn around, and $k_1(\infty) = k_2(\infty) = 0$.

(c) The curve EH is the collection of all $(k_1(0), k_2(0))$ such that $k_2(0) = \left(\frac{k_2^T}{k_1^T}\right)k_1(0)$, $(k_1(0) \geq k_1^T)$ and hence if the $u^* = 0 = v^*$ policy is followed, $(k_1(t), k_2(t))$ is bound to pass through (k_1^T, k_2^T) .

Region I is defined by all points $(k_1(0), k_2(0))$ to the left of the combined curve DEF.

Region II is defined by all points $(k_1(0), k_2(0))$ to the right of DE but above EH.

Region III is defined by all points to the right of EF but below EH.

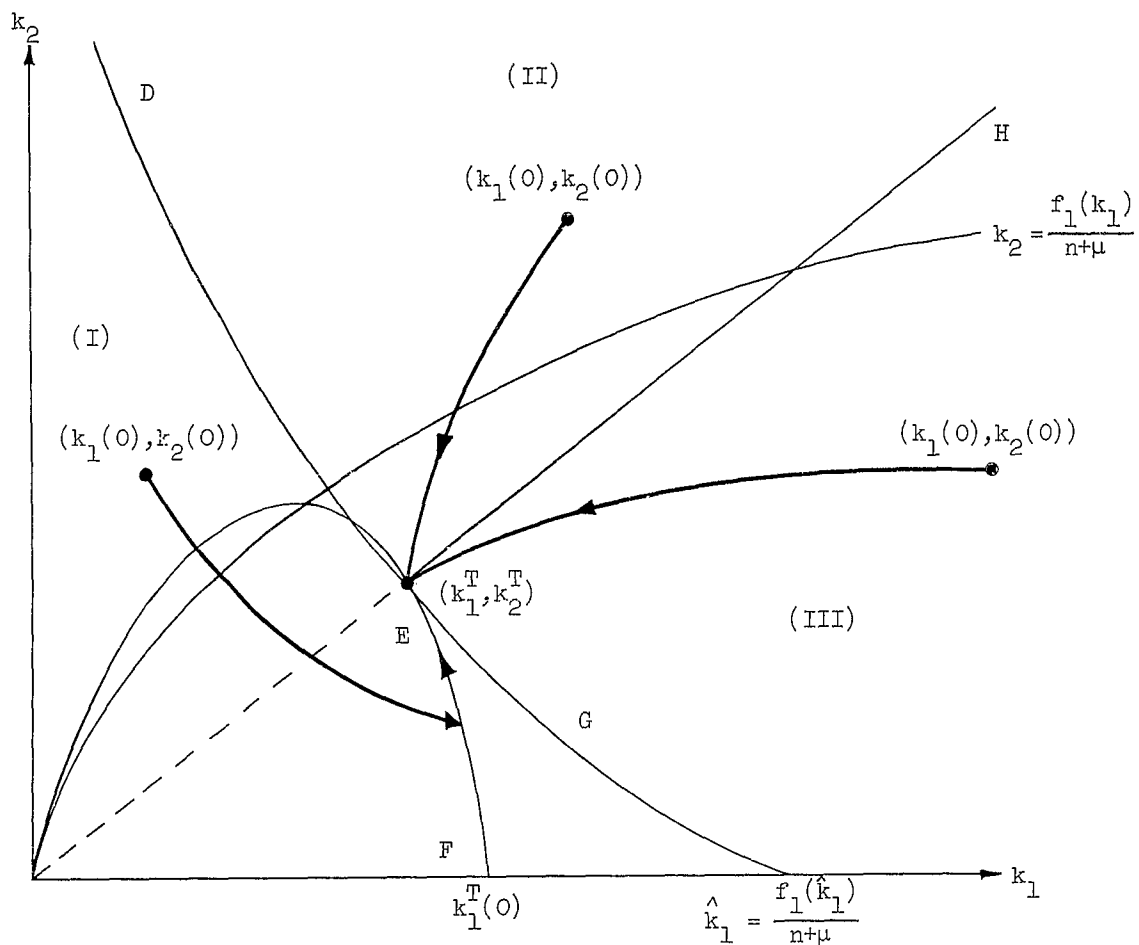


Diagram 2

Optimal Policy for Points in Region I.

It is proved in C.2 below that the optimal policy for $(k_1(0), k_2(0))$ in region I is as follows: follow the $u^* = 1 = v^*$ policy for some time period $(0, t^{**})$ so that k_2 is falling and k_1 is rising until you reach the curve EF. At that moment the policy is changed to $u^* = 0$ $v^* = 1$ and k_1 is falling while k_2 is rising until you reach (k_1^T, k_2^T) .

This strategy of development requires for the initial stage, to build only capital goods and to reinvest them in the machine building industries. Later on as a critical moment is reached, no more capital goods are being invested in the machine building sector and all new capital goods are invested in consumer goods industries. Note that at the "switching" point the capital goods industries have more capital than needed for the optimum terminal path value of k_1^T , but during the next period of transition a certain portion of it is being depreciated.

It is interesting to note that region I does not contain "poor" countries in the conventional sense only. It contains also economies which start off with basically unbalanced capital stock: If an economy has too much (relative to the optimal terminal path) capital in the consumer goods industries and too little capital in the capital goods industries, it still follows the basic policy of first "tipping the balance" in favor of the capital goods sectors and only then approaching the goal of the maximum terminal path level.

In our analysis so far we have not imposed a "floor" on consumption per capita. Consequently, the total "food" production during the transitional period is equal to $\lim_{v \rightarrow 1} (1-v)f_2\left(\frac{k_2(t)}{(1-v)}\right)$. This obviously is unrealistic strategy. The analysis does, however, suggest the basic policy. In imposing floors, we obtain the following results: (a) If we require a certain proportion of the labor force to be employed during the period of transition in the consumer goods sector, then our above analysis remains completely unchanged except that the minimum time is increased. (b) If we require minimum consumption per capita

during the period of transition, then our analysis remains unchanged except for a special region of the (k_1, k_2) diagram which constitutes a trap from which the economy cannot get out and, in fact, except for a special set of conditions the economy will decay completely.

Optimal Policy in Region II.

If the economy starts off in region II, it is capable of going directly from $(k_1(0), k_2(0))$ to (k_1^T, k_2^T) . During the transitional period $u^* = 1$ so that all new capital goods are invested in the capital goods sector and, hence, the capital goods in the consumer goods sector simply depreciate. It is interesting, however, to note that $k_1(t)$ does not have to rise; in fact, it will decline in most cases since the total output of new capital goods does not have to be large so that $k_1(t)$ may grow at a rate smaller than n , the rate of growth of population. It is shown below that there is a unique constant v^* such that the economy will move from $(k_1(0), k_2(0))$ to (k_1^T, k_2^T) . Along the lines HE and DE the extreme policies will be followed. Along EH: $v^* = 0$ and along DE: $v^* = 1$.

Optimal Policy in Region III.

In this region the opposite of II occurs. Here $u^* = 0$, capital in the capital goods sector is allowed to depreciate while the capital stock in the consumer goods sector absorbs all of the newly built capital goods. The amount of new capital goods to be built is determined by v^* which is a unique constant depending upon the initial conditions. Along EH: $v^* = 0$ and along FE: $v^* = 1$. Note here again that k_2 may rise or fall during the transitional period

depending upon whether the rate at which new capital goods are added exceeds or falls short of the rate of growth of population and the rate of depreciation in the consumer goods sector.

(C.2) Rigorous Study of the Optimal Policies

In order to prove the optimality of the policies described in C.1, we have to study first the three "extreme" policies $u^* = v^* = 1$, $u^* = 0$ $v^* = 1$, $u^* = 0$, $v^* = 0$.

C.2.1 The $u^* = 0$ $v^* = 1$ Policy.

Under this policy

$$(10) \quad \begin{aligned} \dot{k}_1 &= -(n+\mu)k_1 \\ \dot{k}_2 &= f_1(k_1) - (n+\mu)k_2 \end{aligned}$$

hence,

$$(11) \quad \begin{aligned} \lim_{t \rightarrow \infty} k_1(t) &= 0 & \lim_{t \rightarrow \infty} k_2(t) &= 0 \\ \dot{k}_2(t) &\leq 0 & \text{if } k_2 &\geq \frac{f_1(k_1)}{n+\mu} \end{aligned}$$

Solving (10) we obtain

$$(12) \quad \begin{aligned} k_1(t) &= k_1(0)e^{-(n+\mu)t} \\ k_2(t) &= e^{-(n+\mu)t} \left[\int_0^t e^{(n+\mu)\tau} f_1(k_1(0)e^{-(n+\mu)\tau}) d\tau + k_2(0) \right] \end{aligned}$$

for $k_2(0) > \frac{f_1(k_1(0))}{n+\mu}$, we have $\dot{k}_2(t) < 0$ $\dot{k}_1(t) < 0$ and $\lim_{t \rightarrow \infty} k_2(t) = \lim_{t \rightarrow \infty} k_1(t) = 0$.

Now insert $k_1(t)$ into $k_2(t)$ in (12) and change variables to obtain

$$(13) \quad k_2 = \frac{k_1}{n+\mu} \int_{k_1}^{k_1(0)} \frac{f_1(x)}{x^2} dx + k_1 \frac{k_2(0)}{k_1(0)},$$

hence, in the (k_1, k_2) plane $k_2 = g(k_1, k_1(0), k_2(0))$. If we select $k_2(0) = 0$, we have a family $k_2 = g(k_1, k_1(0))$ which depends upon

$k_1(0)$ alone. It is clear that for $k_2(0) = 0$ $\frac{dk_2}{dk_1(0)} > 0$ and

$$\sup_{k_1(0)} g(k_1, k_1(0)) = \frac{k_1}{n+\mu} \int_{k_1}^{\infty} \frac{f_1(x)}{x^2} dx.$$

Lemma 1: Let (\bar{k}_1, \bar{k}_2) be arbitrary values with $\bar{k}_2 < \frac{f_1(\bar{k}_1)}{n+\mu}$. Let

$k_2 = g(k_1, k_1(0))$ for some $k_1(0)$. Then

(1) $g(k_1, k_1(0))$ is concave in k_1 (for fixed $k_1(0)$).

(2) $g(k_1, k_1(0))$ is concave in $k_1(0)$ (for fixed k_1).

(3) There exists a unique $\bar{k}_1(0)$ and \bar{t} such that

$$g(k_1(\bar{t}), \bar{k}_1(0)) = k_2 \quad k_1(\bar{t}) = \bar{k}_1.$$

Proof: (1) and (2) follow by direct differentiation of (13).

$$\text{Let } z(k_1, k_1(0)) = \frac{k_1}{n+\mu} \int_{k_1}^{k_1(0)} \frac{f_1(x)}{x^2} dx$$

$$(14) \quad z(\bar{k}_1, \bar{k}_1) = 0 \quad \frac{\partial z}{\partial k_1(0)} > 0 \quad k_1 \leq k_1(0) \leq \infty.$$

We want to prove that there exists a unique $\bar{k}_1(0)$ such that

$$z(\bar{k}_1, \bar{k}_1(0)) = \bar{k}_2.$$

Note that $\frac{f_1(x)}{x^2} > \frac{f_1(\bar{k}_1)}{\bar{k}_1^2}$ all $x > \bar{k}_1$, hence

$$\int_{\bar{k}_1}^{\infty} \frac{f_1(x)}{x^2} dx > \int_{\bar{k}_1}^{\infty} \frac{f_1(\bar{k}_1)}{x^2} dx = \frac{f_1(\bar{k}_1)}{\bar{k}_1}$$

hence

$$z(\bar{k}_1, \infty) = \frac{\bar{k}_1}{n+\mu} \int_{\bar{k}_1}^{\infty} \frac{f_1(x)}{x^2} dx > \frac{f(\bar{k}_1)}{n+\mu} .$$

Since by assumption $\frac{f(\bar{k}_1)}{n+\mu} > \bar{k}_2 > 0$, it follows

$$(14a) \quad z(\bar{k}_1, \infty) > \bar{k}_2 .$$

By (14) and (14a) it follows that there exists a unique $\bar{k}_1(0)$ such that $\bar{k}_1 < \bar{k}_1(0) < \infty$ and

$$z(\bar{k}_1, \bar{k}_1(0)) = \bar{k}_2 ;$$

or:

$$g(\bar{k}_1, \bar{k}_1(0)) = \frac{\bar{k}_1}{n+\mu} \int_{\bar{k}_1}^{\bar{k}_1(0)} \frac{f_1(x)}{x^2} dx = \bar{k}_2 .$$

\bar{t} is determined trivially from $\bar{k}_1 = \bar{k}_2(0)e^{-(n+\mu)\bar{t}}$.

Lemma 1 proves then that as far as the $u^* = 0 \ v^* = 1$ policy is concerned, by studying all $k_2(t)$ solutions with $k_2(0) = 0$ we are in fact studying all $k_2(t)$ with $k_2(0) < \frac{f_1(k_1(0))}{n+\mu}$.

Corollary: There exists a unique concave (in k_1) $g^T(k_1, k_2^T(0))$ such that

$$k_2^T = g^T(k_1^T, k_1^T(0)) \quad \text{and} \quad g^T(k_1^T(0), k_1^T(0)) = 0 .$$

Proof: Since by Theorem 1 $k_2^T < \frac{f_1(k_1^T)}{n+\mu}$, Lemma 1 is applicable.

From Theorem 1, Lemma 1 and the Corollary, it follows that for $u^* = 0 \ v^* = 1$ and all $k_2(0) < \frac{f_1(k_1(0))}{n+\mu}$ the following diagram (Diagram 3) describes the behavior of $(k_1(t), k_2(t))$. For simplicity of notation, we shall write $g^T(k_1)$ instead of $g^T(k_1, k_1^T(0))$.

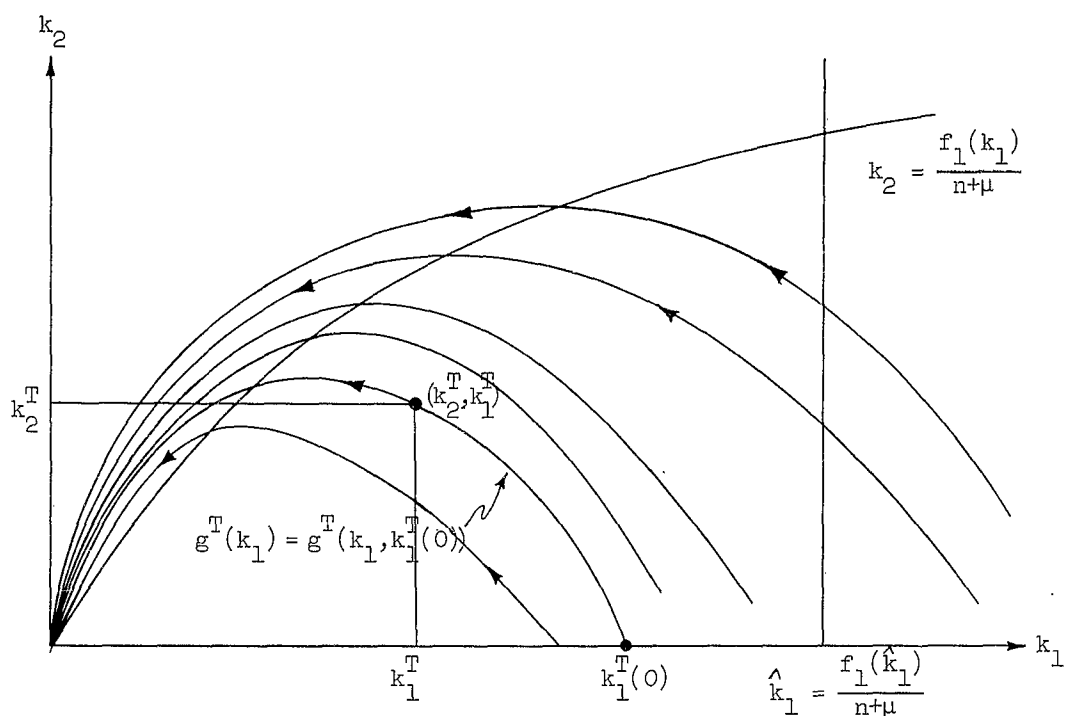


Diagram 3

C.2.2 The $u^* = v^* = 1$ Policy.

Under this policy

$$(15) \quad \begin{aligned} \dot{k}_2 &= -(n+\mu)k_2 \\ \dot{k}_1 &= f_1(k_1) - (n+\mu)k_1. \end{aligned}$$

Here again we can solve (15) and transform it to the (k_1, k_2) plane to obtain a whole family of curves $k_2 = h(k_1, k_1(0))$.

Lemma 2: Under $u^* = v^* = 1$ policy we have

$$(a) \quad \frac{dk_2}{dk_1} > 0 \quad \text{if} \quad k_1 > \frac{f_1(k_1)}{n+\mu}$$

$$(b) \quad \frac{dk_2}{dk_1} < 0 \quad \text{if} \quad k_1 < \frac{f_1(k_1)}{n+\mu}$$

$$(c) \quad \lim_{t \rightarrow \infty} k_2(t) = 0 \quad \lim_{t \rightarrow \infty} k_1(t) = \hat{k}_1 \quad \text{where} \quad \hat{k}_1 = \frac{f_1(\hat{k}_1)}{n+\mu}$$

$$(d) \quad h(k_1, k_1(0)) \text{ is convex in } k_1.$$

Proof: From (15) we have

$$\frac{dk_2}{dk_1} = - \frac{k_2}{k_1 - \frac{f_1(k_1)}{n+\mu}}$$

and

$$\frac{d^2 k_2}{dk_1^2} = \frac{k_2 f_1'(k_1)}{(n+\mu) \left[k_1 - \frac{f_1(k_1)}{n+\mu} \right]^2} > 0$$

and (a) (b) and (d) follow. (c) is seen directly from (15).

Corollary 1: For a fixed $k_1(0)$ $\lim_{k_1 \rightarrow 0} h(k_1, k_1(0)) = \infty$.

Corollary 2: There exists a unique convex $h^T(k_1)$ such that $h^T(k_1^T) = k_2^T$.

In diagrammatic terms, the family of all $h(k_1, k_1(0))$ curves is described as follows:

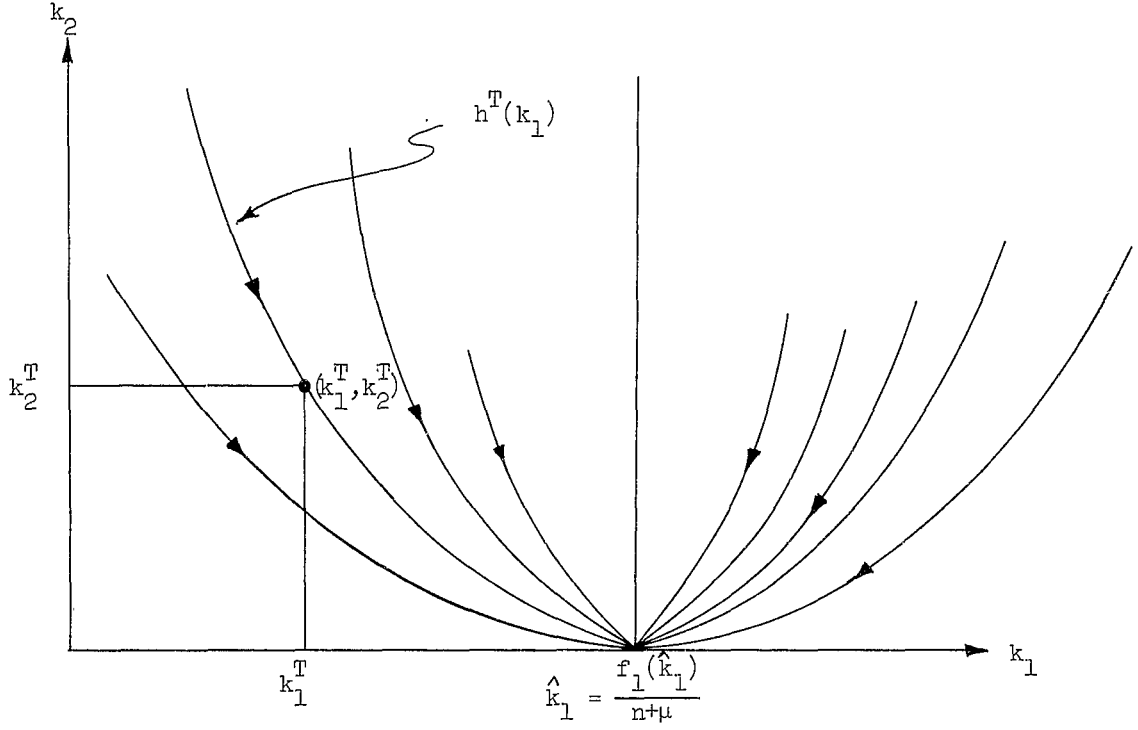


Diagram 4

with the arrows indicating the direction of movement.

The crucial question becomes now: What is the relation between the curve $h^T(k_1)$ under the $u^* = v^* = 1$ policy and $g^T(k_1)$ under the $u^* = 0$ $v^* = 1$ policy.

Theorem 2: Let \hat{k}_1 be the solution of $\hat{k}_1 = \frac{f_1(\hat{k}_1)}{n+\mu}$. Then over the interval $k_1^T < k_1 < \hat{k}_1$ it is true that $g^T(k_1) < h^T(k_1)$. Moreover $k_1^T(0) < \hat{k}_1$.

Proof: Since by Lemma 1 and Lemma 2 g^T is concave in k_1 and h^T is convex in k_1 and since at k_1^T , $h^T(k_1^T) = g^T(k_1^T)$ with

$\left. \frac{dh^T}{dk_1} \right|_{k_1^T} < 0$ $\left. \frac{dg^T}{dk_1} \right|_{k_1^T} < 0$ it follows that in order to prove the theorem

it suffices to prove that

$$z = \frac{dg^T(k_1)}{dk_1} \bigg|_{k_1^T} - \frac{dh^T(k_1)}{dk_1} \bigg|_{k_1^T} < 0 .$$

From Lemma 1 and Lemma 2 we know, however, that

$$z = \frac{k_2^T - \frac{f_1(k_1^T)}{n+\mu}}{k_1^T} - \frac{k_2^T}{k_1^T - \frac{f_1(k_1^T)}{n+\mu}} = \frac{f_1(k_1^T) \left[\frac{f_1(k_1^T)}{n+\mu} - (k_2^T + k_1^T) \right]}{(n+\mu) \left[k_1^T - \frac{f_1(k_1^T)}{n+\mu} \right]} .$$

By Theorem 1 $k_1^T - \frac{f_1(k_1^T)}{n+\mu} < 0$. By the same Theorem 1

$$k_1^T + k_2^T < \frac{f_1(k_1^T)}{n+\mu}$$

which proves that $z < 0$. The result that $k_1^T(0) < \hat{k}_1$ is trivial.

Theorem 3: For any k_1 such that $k_1^T \leq k_1 < k_1^T(0)$
 $f(k_1) - (n+\mu)(k_1 + g^T(k_1)) > 0$.

Proof: Let $z = f(k_1) - (n+\mu)(k_1 + g^T(k_1))$. We know that

$$g^T(k_1) = \frac{k_1}{n+\mu} \int_{k_1}^{k_1^T(0)} \frac{f_1(x)}{x^2} dx .$$

Hence

$$\frac{z}{k_1} = \frac{f(k_1)}{k_1} - (n+\mu) - \int_{k_1}^{k_1^T(0)} \frac{f_1(x)}{x^2} dx .$$

Note, however, that $\int_{k_1}^{k_1^T(0)} \frac{f(x)}{x^2} dx$ is convex in k_1 and

$k_1 \leq k_1^T(0)$. Hence

$$\begin{aligned}
\frac{z}{k_1} &\geq \frac{f(k_1)}{k_1} - (r+\mu) - \frac{f(k_1)}{k_1^2} (k_1 - k_1^T(0)) \\
&= -(r+\mu) + \frac{f(k_1)}{k_1^2} k_1^T(0) \\
&\geq -(r+\mu) + \frac{f(k_1)}{k_1} \quad (\text{since } k_1 \leq k_1^T(0)) \\
&> 0 \quad \text{since } k_1 < \hat{k}_1.
\end{aligned}$$

Hence $z > 0$.

Theorem 4: Let $k_2 = h(k_1, k_1(0))$ be an arbitrary ($v^* = 1 = u^*$ policy) function with $h(k_1(0), k_1(0)) < h^T(k_1(0))$. Then there exists a unique pair (k_1^{**}, k_2^{**}) such that $h(k_1^{**}, k_1(0)) = g^T(k_1^{**})$.

Proof: First note that $h(k_1(0), k_1(0)) < h^T(k_1(0)) \Rightarrow h(k_1, k_1(0)) < h^T(k_1)$ for all k_1 . Now form the function

$$W(k_1) = \begin{cases} h^T(k_1) - h(k_1, k_1(0)) & k_1 \leq k_1^T \\ g^T(k_1) - h(k_1, k_1(0)) & k_1^T \leq k_1 \leq k_1^T(0) \end{cases}.$$

Clearly $W(k_1) > 0$ for $k_1 \leq k_1^T$

$$W(k_1 = k_1^T(0)) = g^T(k_1^T(0), k_1^T(0)) - h(k_1^T(0), k_1(0)).$$

But $g^T(k_1^T(0), k_1^T(0)) = 0$ and $h(k_1^T(0), k_1(0)) > 0$ since $h(k_1, k_1(0)) = 0$ for $k_1 = \hat{k}_1$ only. And $\hat{k}_1 > k_1^T(0)$. Hence $W(k_1 = k_1^T(0)) < 0$. Hence there exist a k_1^{**} such that $W(k_1^{**}) = 0$, hence $h(k_1^{**}, k_1(0)) = g^T(k_1^{**})$. Note, however, that $W(k_1)$ is monotone, hence k_1^{**} is unique.

C.2.3 The $u^* = v^* = 0$ Policy.

In this case

$$(16) \quad \begin{aligned} \dot{k}_1 &= -(n+\mu)k_1 \\ \dot{k}_2 &= -(n+\mu)k_2 \end{aligned}$$

Hence $k_2 = \frac{k_2(0)}{k_1(0)} k_1$ is the description of the $u^* = v^* = 0$ policy in the (k_1, k_2) plane.

$k_2 = \frac{k_2(0)}{k_1(0)} k_1$ are straight lines and obviously there is only one such line that goes through the point (k_1^T, k_2^T) , namely $k_2 = \frac{k_2^T}{k_1^T} k_1$.

We have completed the preparation of all the material needed for the proof of the optimal policy. We turn finally to this proof.

C.2.4 Proof of the Optimal Policy.

(a) Region I

Let $(k_1(0), k_2(0))$ be arbitrary initial conditions in region 1. If $k_2(0) = h^T(k_1(0), k_1(0))$ ($k_1(0) \leq k_1^T$) or $k_2(0) = g^T(k_1(0))$ $k_1^T \leq k_1(0) \leq k_1^T(0)$ the optimal policy will be to follow these curves respectively ($u^* = v^* = 1$ in the first, and $u^* = 0$ $v^* = 1$ in the second). If $(k_1(0), k_2(0))$ is interior point of region 1, it has the following features: (1) If the $v^* = 1$ $u^* = 0$ policy is followed $k_1(t)$ and $k_2(t)$ will tend to 0. (2) If $u^* = 0$ $v^* = 0$ policy is followed $k_1(t)$ $k_2(t)$ will tend to 0. (3) If $u^* = 1 = v^*$ is followed, then there exist (by Theorem 4) a t^{**} and k_1^{**} such that $h(k_1^{**}, k_1(0)) = g^T(k_1^{**})$.

Theorem 5: If $(k_1(0), k_2(0))$ belong to region I, the optimal policy is:

$$\begin{aligned} u^*(t) &= v^*(t) = 1 & 0 \leq t \leq t^{**} \\ u^*(t) &= 0 \quad v^*(t) = 1 & t^{**} \leq t \leq T. \end{aligned}$$

Proof: (1) Under $u^*(t) = v^*(t) = 1$

$$\psi_2(t) = Ae^{(n+\mu)t}, \quad \psi_1(t) = e^{-\int_0^t [f_1' - (n+\mu)] d\tau} [C]$$

$$\dot{k}_2(t) = -(n+\mu)k_2, \quad \dot{k}_1 = f_1(k_1) - (n+\mu)k_1.$$

Choose $A > 0$ and $C = Ae^{\int_0^{t^{**}} f_1' d\tau}$. Hence

$$\begin{aligned} \psi_2(t) &> 0 \quad \psi_1(t) > 0 \quad \text{and} \quad \psi_1(t) - \psi_2(t) > 0 \quad \text{for} \quad 0 \leq t < t^{**} \\ \psi_1(t^{**}) - \psi_2(t^{**}) &= 0. \end{aligned}$$

(2) Under $u^*(t) = 0 \quad v^*(t) = 1 \quad t^{**} \leq t \leq T$.

$$\psi_2(t) = Ae^{(n+\mu)t}, \quad \psi_1(t) = e^{(n+\mu)t} [A - A \int_{t^{**}}^t f_1' dx]$$

and $\dot{k}_2(t) = f_1(k_1) - (n+\mu)k_2 \quad \dot{k}_1(t) = -(n+\mu)k_1(t)$. Hence:

$$\psi_2(t) > 0 \quad \psi_2(t^{**}) = \psi_1(t^{**}) \quad \psi_2(t) - \psi_1(t) > 0 \quad \text{for} \quad t^{**} < t < T.$$

(3) The choice of $\psi_1(t)$ and $\psi_2(t)$ for $0 \leq t \leq T$ is clearly a choice of continuous functions, they are also solutions of the differential equations (8a-8b).

$$\begin{aligned} (4) \quad H &= \psi_1[uvf_1(\frac{k_1}{v}) - (n+\mu)k_1] + \psi_2[(1-u)vf_1(\frac{k_1}{v}) - (n+\mu)k_2] \\ &= (\psi_1 - \psi_2)[uvf_1(\frac{k_1}{v})] - (n+\mu)[\psi_1(t)k_1(t) + \psi_2(t)k_2(t)] \\ &\quad + \psi_2(t)vf_1(\frac{k_1}{v}). \end{aligned}$$

Hence when $\psi_1 > 0$ $\psi_2 > 0$ $\psi_1 > \psi_2$

$$\sup_{u,v} H = (\psi_1 - \psi_2)[f_1(k_1)] - (n+\mu)[\psi_1(t)k_1(t) + \psi_2(t)k_2(t)] + \psi_2(t)f_1(k_1) .$$

Hence

$$u^* = 1 = v^* .$$

when $\psi_2 > 0$ but $\psi_2(t) > \psi_1(t)$

$$\sup_{u,v} H = -(n+\mu)[\psi_1(t)k_1(t) + \psi_2(t)k_2(t)] + \psi_2(t)f_1(k_1) .$$

Hence

$$u^* = 0 \quad v^* = 1 .$$

This establishes the (I) part of the consistency conditions of the Pontriagin Theorem. We have to prove now that $\sup_{v,u} H \geq 0$ for all $0 \leq t \leq T$.

(5) Over the interval $0 \leq t \leq t^{**}$,

$$(17) \quad H = Ae^{\int_0^{t^{**}} f_1' d\tau} e^{-\int_0^t [f_1' - (n+\mu)] d\tau} [f_1(k_1) - (n+\mu)k_1] + Ae^{(n+\mu)t} [-(n+\mu)k_2(t)] .$$

From (15) we have

$$(18) \quad k_2(t) = k_2(0)e^{-(n+\mu)t} ;$$

hence,

$$(18a) \quad k_2(0) = k_2^{**} e^{(n+\mu)t^{**}} .$$

Also, since $\frac{dk_1}{dt} = f_1(k_1) - (n+\mu)k_1$,

$$(19) \quad dt = \frac{dk_1}{f_1(k_1) - (n+\mu)k_1} .$$

Now,

$$H = A e^{\int_0^{t^{**}} f_1' d\tau - [\int_0^{t^{**}} [f_1' - (n+\mu)] d\tau - \int_t^{t^{**}} [f_1' - (n+\mu)] d\tau} [f_1(k_1) - (n+\mu)k_1] \\ - A e^{(n+\mu)t} (n+\mu)k_2(0) e^{-(n+\mu)t};$$

hence,

$$H = A e^{(n+\mu)t^{**}} [f_1(k_1) - (n+\mu)k_1] e^{\int_t^{t^{**}} [f_1' - (n+\mu)] d\tau} - A (n+\mu)k_2(0).$$

Note now that by (19) (change of variables),

$$(20) \quad \int_t^{t^{**}} [f_1' - (n+\mu)] d\tau = \int_{k_1}^{k_1^{**}} \frac{f_1' - (n+\mu)}{f_1(x) - (n+\mu)x} dx = \log(f_1(x) - (n+\mu)x) \Big|_{k_1}^{k_1^{**}}.$$

Now, using (18a) to substitute for $k_2(0)$, and (20), we obtain

$$H = A e^{(n+\mu)t^{**}} [f_1(k_1) - (n+\mu)k_1] \frac{f_1(k_1^{**}) - (n+\mu)k_1^{**}}{f_1(k_1) - (n+\mu)k_1} - A (n+\mu)k_2^{**} e^{(n+\mu)t^{**}}$$

or

$$H = A e^{(n+\mu)t^{**}} [f_1(k_1^{**}) - (n+\mu)k_1^{**} - (n+\mu)k_2^{**}].$$

Noting, however, that $k_2^{**} = g^T(k_1^{**})$, and using Theorem 3, we conclude that $H > 0$ and a constant.

(6) Over the interval $t^{**} \leq t \leq T$

$$H = e^{(n+\mu)t} [A - A \int_{t^{**}}^t f_1' dx] (n+\mu)k_1 + A e^{(n+\mu)t} [f_1(k_1) - (n+\mu)k_2] \\ = A e^{(n+\mu)t} [f_1(k_1) - (n+\mu)(k_2 + k_1)] + A e^{(n+\mu)t} (n+\mu)k_1 \int_{t^{**}}^t f_1' dx.$$

Since $k_2 = g^T(k_1)$, Theorem 3 is again applicable and $H > 0$.

To check that H is a constant let

$$\frac{H}{Ae^{(n+\mu)t}} = z(t) = f_1(k_1) - (n+\mu)(k_1 + k_2) + (n+\mu)k_1 \int_{t^{**}}^t f_1' dx .$$

Since $\dot{k}_1 = -(n+\mu)k_1$ $\dot{k}_2 = f_1(k_1) - (n+\mu)k_2$

$$z(t) = -(n+\mu)[f_1(k_1) - (n+\mu)(k_1 + k_2) + (n+\mu)k_1 \int_{t^{**}}^t f_1' dx] ,$$

hence

$$z(t) = z(T)e^{-(n+\mu)(t-T)}$$

hence

$$H = Ae^{(n+\mu)t} z(T)e^{-(n+\mu)(t-T)}$$

hence

$$H = Az(T)e^{(n+\mu)T}$$

a constant where

$$z(T) = f_1(k_1^T) - (n+\mu)(k_1^T + k_2^T) + (n+\mu)k_1^T \int_{t^{**}}^T f_1' dx .$$

Region II

Region II is characterized by the following conditions:

$$(a) \quad k_2(0) \geq \frac{k_2^T}{k_1^T} k_1(0)$$

$$(b) \quad k_2(0) \geq h^T(k_1(0)) .$$

Theorem 6: If $(k_1(0), k_2(0))$ belong to region II, the optimal policy is $u^* = 1$ and v^* is to be determined by the initial conditions. There exists a unique constant v^* such that $k_1(T) = k_1^T$ $k_2(T) = k_2^T$.

Proof: Choose $\psi_1(t) = 0$ $\psi_2(t) = Ae^{(n+\mu)t}$ $A < 0$.

$$H = Ae^{(n+\mu)t} [(1-u)v f_1\left(\frac{k_1}{v}\right) - (n+\mu)k_2]$$

and

$$\sup_{u,v} H = -(n+\mu)Ae^{(n+\mu)t} k_2(t) \quad \text{with } u^* = 1.$$

Under $u^* = 1$

$$\dot{k}_2(t) = -(n+\mu)k_2$$

hence

$$k_2(t) = k_2(0)e^{-(n+\mu)t}$$

hence

$$\begin{aligned} \sup_{u,v} H &= -A(n+\mu)e^{+(n+\mu)t} k_2(0)e^{-(n+\mu)t} \\ &= -A(n+\mu)k_2(0) > 0 \quad \text{since } A < 0. \end{aligned}$$

Since $k_2(0) \geq k_2^T$, there exists a T^* such that $k_2^T = k_2(0)e^{-(n+\mu)T^*}$.

All the conditions required by the maximum principle are satisfied

except that we have to show the existence of a v^* . To do this, notice that

$$\dot{k}_1 = v f_1\left(\frac{k_1}{v}\right) - (n+\mu)k_1$$

has a solution $k_1 = m(t, k_1(0), v)$ where v is a constant and m is continuous. By the definition of region II, however:

$$\begin{aligned} \text{If } v = 0 \quad \text{then } k_2(T) &= k_2^T \\ &\text{but } k_1(T) \leq k_1^T \\ \text{If } v = 1 \quad \text{then } k_2(T) &= k_2^T \\ &\text{but } k_1(T) \geq k_1^T. \end{aligned}$$

(If $k_2(0) = \frac{k_2^T}{k_1^T} k_1(0)$ then $v^* = 0$ is optimal, if $k_2(0) = h^T(k_1(0))$

then $v^* = 1$ is optimal.) Since $m(t, k_1(0), v)$ is continuous, there must exist a $0 \leq v^* \leq 1$ such that $k_1(T) = m(T, k_1(0), v^*) = k_1^T$.

Region III

Region III is characterized by the following conditions:

$$(a) \quad k_2(0) \leq \frac{k_2^T}{k_1^T} k_1(0)$$

$$(b) \quad k_2(0) \geq g^T(k_1(0)) .$$

Theorem 7: If $(k_1(0), k_2(0))$ belong to region III, the optimal policy is $u^* = 0$ and v^* is to be determined by the initial conditions. There exist a unique constant v^* such that $k_1(T) = k_1^T$ $k_2(T) = k_2^T$.

The proof is omitted because of its similarity to the proof of Theorem 6. We may only note that in this case $\psi_2 = 0$ ($A = 0$) $\psi_1 = Be^{(n+\mu)t}$ $B < 0$ with the resulting $u^*(t) = 0$.

(C.3) An Evaluation of the Results

In order to evaluate our results, let us consider again the entire $(k_1(0), k_2(0))$ plane and interpret the nature of the initial conditions using the following diagram (see Diagram 5):

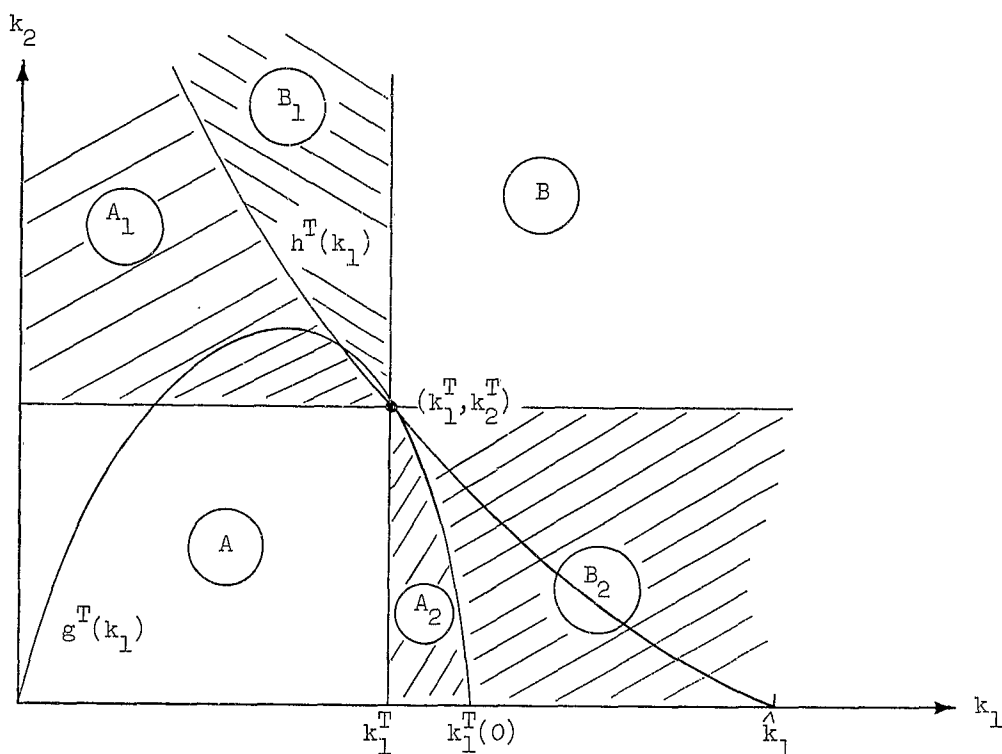


Diagram 5

Regions A and B have a clear cut interpretation. Countries in region A are "truly" underdeveloped in the sense that they have both k_1 and k_2 below the von Neumann target. Countries in region B are truly "overdeveloped" in the sense that their k_1 and k_2 are both above the target. Economies in any other region can be interpreted to be in "unbalanced" state: Only one of the sectors has a capital labor ratio (note: $k_1 = \frac{k_1}{L}$!) above the target, while the other is below the target.

It is quite clear that the criterion of minimum time is not very meaningful for economies in region B. This is demonstrated mathematically by the indeterminacy of the v policy for this region. In other words, the truly minimum time solution for economies starting in region B is to throw away capital and thus reduce $k_1(0)$ and

$k_2(0)$ to k_1^T and k_2^T . We did not allow this since the path has to satisfy the differential equations (4a-4b). But surely, among all the paths between the initial conditions and (k_1^T, k_2^T) we want to choose the one that maximizes some utility function of consumption. In this sense, we would like to stay in region B as long as we can!

Another feature of our solution is important. The unrestricted minimum time solution for economies in regions A requires consumption per capita to be zero during the transition period. This is part of the results obtained from the criterion of maximum speed. This result is not as unrealistic as it may seem since we can always impose floors on consumption or on the proportion of the labor force employed in the consumer goods industries. This feature of low consumption per capita has appeared in many papers (including some in which an explicit utility function was maximized) and was resolved by imposing such artificial floor based on some myth called "subsistence." This is certainly meaningless.

The discussion points out the need for some kind of social preference function that would lead us to stay in region B (Diagram 5) as long as we can on the one hand, and prevent consumption from falling to 0 in regions type A, on the other. Floors should come as a consequence of some general principles of choice over time rather than an arbitrary decision by the investigator. Unfortunately, such preference functions as the discounted sum of consumption (or consumption per capita) do not prevent consumption from falling to zero during the transition; thus, Uzawa [16] had to resort to the artificial consumption floor to remedy the situation.

The principle of maximum speed analyzed in this paper has the great merit of exposing the basic features of optimal strategies with respect to objective functions that regard rapid development as a desirable result. It also demonstrates the similarity in development strategy between economies of entirely different nature. For example, let us consider Germany in 1945; most of the capital goods industries were partly destroyed, while the consumer goods industries like agriculture, were much less damaged. Since there was a general reduction in the labor force due to the war, Germany may have found itself in region A_1 . The basic features of the German recovery were to build very rapidly the capital goods industries while keeping consumption restricted by the agreement of the unions to maintain low wages. These are the main characteristics of the strategy resulting from the maximum speed principle. It seems to us quite plausible that the same principle may have guided the Russian strategy of economic development; moreover, it may be that the principle of maximum speed is the principle that underlies the development strategy of many underdeveloped countries today.

(D) A Comment

We have noted in the introduction that we regard this paper as a continuation of Srinivasan's work [12]. In his paper, Srinivasan worked with a two sector model where each sector was represented by an "activity analysis" model of production. Srinivasan was unable to obtain a general solution to the minimum time problem. Consequently, he considered only the class of paths of the type (in our notation)

$$\begin{array}{lll} u^* = 1 & v^* = 1 & 0 \leq t \leq t^{**} \\ u^* = 0 & v^* = 1 & 0 \leq t \leq T \end{array}$$

and within this class he chose the one with minimum T (see [12], pages 81-87). In this way, his problem was only the determination of t^{**} . Moreover, Srinivasan considered initial conditions in region A (diagram above) only. In view of these facts, we regard our work as a generalization of the Srinivasan solution to all feasible paths starting from any initial conditions.

FOOTNOTES

- 1/ This last point was raised in [7] and [8].
- 2/ The form of the theorem (II) follows from the fact that both (4) and (8a-8b) have solutions. This is the result of assumptions (1a) and (1b). Instead of proving this fact, we shall later on simply obtain the explicit solutions and use them in the analysis.
- 3/ From (8a-8b) it is clear that $\psi_1 = \psi_2 \neq 0$ over an interval $t_0 \leq t \leq t_1$ immediately imply $u^*(t) = v^*(t) = 0$ over this interval.

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